Geα -CLOSED SETS IN TOPOLOGICAL SPACES

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Abstract

In this paper, we introduce the notion of egα spaces and investigate some of their properties. -closed sets in topological

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Introduction and Preliminaries

Levine [6,7] introduced the concept of generalized closed sets and semi-closed sets in topological spaces. Maki et al. introduced generalized α-closed sets (briefly gα-closed sets) [9] and α-generalized closed sets (briefly αg-closed sets) [8]. The concept of g-closed sets [16,17], *g-closed sets [14] and] gs-closed sets b [15] are introduced by M.K.R.S. Veera Kumar. In this paper, we introduce a new class of sets, namely, gα-closed sets and present some of its properties.

Throughout this paper (X, τ), (Y, σ) and (Z, η) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A) and int(A) denote the closure of A and the interior of A respectively. P (X) denotes the power set of X . We recall the following definitions, which are useful in the sequel.

Definition 1.1. A subset A of a space (X, τ) is called

1. a pre-open set [10] if A ⊆ int(cl(A)) and a pre-closed set if cl(int(A)) ⊆ A,
2. a semi-open set [7] if A ⊆ cl(int(A)) and a semi-closed set [7] if int(cl(A)) ⊆ A,
3. an α-open set [11] if A ⊆ int(cl(int(A))) and an α-closed set [11] if cl(int(cl(A))) ⊆ A,
4. a semi-preopen set [1] if A ⊆ cl(int(cl(A))) and a semi-preclosed set [1] if

1. int(cl(int(A))) ⊆ A and

5. a regular open set if A = int(cl(A)) and a regular closed set if cl(int(A)) = A.

The pre-closure (resp. semi-closure, α-closure, semi-preclosure) of a subset A of a space (X, τ) is the intersection of all pre-closed (resp. semi-closed, α-closed, semi-preclosed) sets that contain A and is denoted by pcl(A) (resp. scl(A), αcl(A), spcl(A)).

Definition 1.2. A subset A of a space (X, τ) is called a

1. a generalized closed (briefly g-closed) set [6] if cl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ); the complement of a g-closed set is called a g-open set,
2. a semi-generalized closed (briefly sg-closed) set [3] if scl(A) ⊆ U whenever A ⊆ U and U is semi-open in (X, τ),

3. a generalized semi-closed (briefly gs-closed) set [2] if scl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ),
4. an α-generalized closed (briefly αg-closed) set [8] if αcl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ),
5. a generalized α-closed (briefly gα-closed) set [9] if αcl(A) ⊆ U whenever
6. A ⊆ U and U is α-open in (X, τ), a gα*-closed set [9] if αcl(A) ⊆ int(U) whenever A ⊆ U and U is α-open in (X, τ),
6. a generalized semi-preclosed (briefly gsp-closed) set [4] if spcl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ),
7. a generalized preregular-closed (briefly gpr-closed) set [5] if pcl(A) ⊆ U Whenever A ⊆ U and U is regular open in (X, τ),
8. a g*-closed set [13] if cl(A) ⊆ U whenever A ⊆ U and U is g-open in (X, τ),
9. a g-closed set [16,17] if cl(A) ⊆ U Whenever A ⊆ U and U is semi-open in g-closed set is called a b g-open set,
10. a *g-closed set [14] if cl(A) ⊆ U Whenever A ⊆ U and U is g-open in (X, τ); the complement of a *g-closed set is called a *g-open set,
7. a]gs-closed set [15] if scl(A) ⊆ U whenever A ⊆ U and U is *g-open in (X, τ); the complement of a]gs-closed set is called a] gs-open set and a gs-closed set [12] if scl(A) ⊆ U whenever A ⊆ U and U is]gs-open in (X, τ).

Notation 1.3 For a Topological Space (X, τ), C (X, τ) (resp. αC (X, τ), GC (X, τ),

SGC (X, τ), GSC (X, τ), αGC (X, τ), GαC (X, τ), Gα*C (X, τ), GSP C (X, τ), GP RC (X, τ), G*C (X, τ), *GC (X, τ),]GSC (X, τ), GeSC (X, τ)) denotes the class of all closed (resp. α-closed, g-closed, sg-closed, gs-closed, αg-closed, gα-closed, gα*-closed, gsp-closed, gpr-closed, g*-closed, *g-closed,]gs-closed, e gs-closed) subsets of (X, τ).

2. gα-closed sets e

We introduce the following definition.

Definition 2.1 A subset A of (X, τ) is called a gα-closed set if αcl(A) ⊆e U whenever A ⊆ U and U is]gs-open in (X, τ).

Theorem 2.2. Every α-closed set is a gα-closed set and thus every closed set is e gα-closed. e

Proof. Let A be an α-closed set in (X, τ), then A = αcl(A). Let A ⊆ U , U is] gs-open in (X, τ). Since A is α-closed, A = αcl(A) ⊆ U . This shows that A is gα-closed set. The second part of the theorem It follows from the fact that every e closed set is α-closed. The converses in the above theorem are not true as can be seen by the follow- ing example.

Example 2.3. Let X = {a, b, c} and τ = {X, φ, {a, b}}. Here αC (X, τ) = {X, φ, {c}} and gα-closed set. G^{e}αC (X, τ) = {X, φ, {c}, {b, c}, {a, c}} and let A = {b, c}. Then A is not an α-closed and thus it is not closed. However A is a e

Thus the class of gα-closed the sets sets properly contains e and closed sets. classes of α-closed

Theorem 2.4.

(a) Every gα-closed set is a gs-closed set and thus gsp-closed and gpr-closed.^{e}

(b) Every gα-closed set is a gα-closed set and thus αg-closed.^{e}

(c) Every gα-closed set is a sg-closed set and thus semi-preclosed.^{e}

Proof. It follows from the definitions.

The following examples show that these implications are not reversible.

Example 2.5. Let X = {a, b, c} and τ = {X, φ, {a}, {b, c}}. Here GSC (X, τ) = P (X), GSP C (X, τ) = P (X), GP RC (X, τ) = P (X) and G^{e}αC (X, τ) = {X, φ, {a}, {b, c}} and let A = {b}. Then A is gs-closed, gsp-closed and gpr-closed. However A is not a gα-closed set.^{e}

Example 2.6. Let X = {a, b, c} and τ = {X, φ, {b}, {b, c}}. Here GαC (X, τ) = {X, φ, {a}, {c}, {a, b}, {a, c}}, αGC (X, τ) = {X, φ, {a}, {c}, {a, b}, {a, c}} and G^{e}αC (X, τ) = {X, φ, {a}, {c}, {a, c}} and let A = {a, b}. Then A is gα-closed and αg-closed. However A is not a gα-closed set.^{e}

Example 2.7. Let X = {a, b, c} and τ = {X, φ, {a}, {b}, {a, b}}. Here SGC (X, τ) = {X, φ, {a}, {b}, {c}, {b, c}, {a, c}}, SP C (X, τ) = {X, φ, {a}, {b}, {c}, {b, c}, {a, c}} and G^{e}αC (X, τ) = {X, φ, {c}, {b, c}, {a, c}} and let A = {a}. Then A is sg-closed and semi-preclosed. However A is not a gα-closed set.^{e}

Theorem 2.8. Every e Proof. It follows from the definitions.

gs-closed set. The converse of the above theorem need not be true by the following example.

Example 2.9. Let X = {a, b, c} and τ = {X, φ, {a}}. Here G^eSC (X, τ) =
{X, φ, {a}, {b}, {c}, {a, b}, {b, c}},
$G^e\alpha C$ (X, τ) = {X, φ, {b}, {c}, {b, c}} and let
gα-closed set.
gs-closed but not a e

Theorem 2.10.

(a) gα-closedness is independent of g-closedness, g^*-closedness and *g-closedness.e
(b) e g-closedness.
(c) gα-closedness is independent of $g\alpha^*$-closedness.e
Proof. It follows from the following examples.

Example 2.11. Let X = {a, b, c} and τ = {X, φ, {a}, {a, c}}. Here GC (X, τ) =
{X, φ, {b}, {a, b}, {b, c}}, G^*C (X, τ) = {X, φ, {b}, {a, b}, {b, c}}, $^*GC$ (X, τ) =
{X, φ, {b}, {a, b}, {b, c}} and $G^e\alpha C$ (X, τ) = {X, φ, {b}.{c}, {b, c}}. Then {a, b} is
gα-closed set and also {c} is gα-closed,
but not even a g-closed, g^*-closed and *g-closed.

Example 2.12. Let X = {a, b, c} and τ = {X, φ, {a}, {a, c}}. Here G^bC (X, τ) = {X, φ, {b}, {b, c}} and
$G^e\alpha C$ (X, τ) = {X, φ, {b}, {c}, {b, c}}. Then {c} is e
closed, but not a g-closed set. b
Let X = {a, b, c} and τ = {X, φ, {a}, {b, c}}. Here G^bC (X, τ) = P (X) and
g-closed, but not a gα-closed set e
Example 2.13. Let X = {a, b, c} and τ = {X, φ, {a}, {b, c}}. Here $G\alpha^*C$ (X, τ) =
P (X) and $G^e\alpha C$ (X, τ) = {X, φ, {a}, {b, c}}. Then {b} is $g\alpha^*$-closed, but not a gα-
closed set.

Let X = {a, b, c} and τ = {X, φ, {a}}. Here $G\alpha^*C$ (X, τ) = {X, φ, {b, c}} and $G^e\alpha C$ (X, τ) =
{X, φ, {b}, {c}, {b, c}}. Then {b} is gα-closed, but not a $g\alpha^*$-closed set.

Theorem 2.14. Let A be a subset of (X, τ).
(a) If A is gα-closed, then αcl(A) A does not contain any non-empty] gs-close^e set.
e

Proof.

gα-closed and A ⊆ B ⊆ αcl(A), then B is gα-closed.
gα-closed and let F be a non-empty]gs-closed set with(a) Suppose that A is F ⊆ αcl(A) − A.
Then A ⊆ X − F and so αcl(A) ⊆ X − F . Hence F ⊆ X − αcl(A), a contradiction.
(b) Let U be a]gs-open set of (X, τ) such that B ⊆ U . Then A ⊆ U . Since A is gα-closed, αcl(A) ⊆ U . Now αcl(B) ⊆ αcl(αcl(A)) ⊆ U . Therefore B is also a gα-closed set of (X, τ).e

Theorem 2.15. Let A and B be subsets of a topological space (X, τ). Then the union of two e gα-closed set in (X, τ).
gα-closed sets. Let A ∪ B ⊆ U , U is] gs-open. Since
Proof. Let A and B be e A and B are gα-closed sets, αcl(A) ⊆ U , αcl(B) ⊆ U . This implies that αcl(A ∪ B) = αcl(A) ∪ αcl(B) ⊆ U , (since τ^{α} =α-open set forms a topology [9]) and so αcl(A ∪ B) ⊆ U . Therefore A ∪ B is gα-closed.e
We need the following notations:
For a subset E of a space (X, τ), we define the following subsets of E.
E_τ = {x ∈ E/{x} ∈ τ };
E_F = {x ∈ E/{x} is closed in (X, τ)};
$E_{g\alpha o}$ = {x ∈ E/{x} is gα-open in (X, τ)};
E] $_{gsc}$ = {x ∈ E/{x} is]gs-closed in (X, τ)}.

Lemma 2.16. For any space (X, τ), X = X] $_{gsc}$ ∪ $X_{e\ g\alpha o}$ holds. Proof. Let x ∈gα X. Suppose that {x} is not]gs-closed set in (X, τ). Then X is an unique]gs-open set containing X − {x}. Thus X − {x} is e -closed in (X, τ) and so { gα-open. Therefore x ∈ X] $_{gsc}$ ∪ X_e holds. x} is e gαo
We need more notations:
For a subset A of (X, τ), ker(A) = ∩{U/U ∈ τ and A ⊆ U };
] GSO-ker(A) = ∩{U/U ∈]GSO(X, τ) and A ⊆ U }.
Theorem 2.17. For a subset A of (X, τ), the following conditions are equiv- alent.
(1) A is gα-closed in (X, τ).e
(2) αcl(A) ⊆] GSO-ker(A) holds.
(3) (i) αcl(A) ∩ X] $_{gsc}$ ⊆ A and (ii) αcl(A) ∩ X] $_{gso}$ ⊆]GSO-ker(A) holds.
Proof. (1) ⇒ (2) Let x ∉] GSO-ker(A). Then there exists a set U ∈] GSO(X, τ) such that x ∉e U and A ⊆ U . Since A is gα-closed, αcl(A) ⊆ U and so x ∉ αcl(A).

This shows that αcl(A) ⊆]GSO-ker(A).
(2) ⇒ (1) Let U ∈] GSO(X, τ) such that A ⊆ U . Then we have that]GSO-ker(A) ⊆ U and so by (2) αcl(A) ⊆ U . Therefore A is gα-closed.

(2) ⇒ (3) (i) First we claim that]GSO-ker(A) ∩ X] $_{gsc}$ ⊆ A. Indeed, let x ∈] GSO-ker(A)∩X] $_{gsc}$ and assume that x ∈/ A. Since the set X −{x} ∈] GSO(X, τ) and A ⊆ X − {x},] GSO-ker(A) ⊆ X − {x}. Then we have that x ∈ X − {x} and so this is a contradiction. Thus we show that] GSO-ker(A) ∩ X] $_{gsc}$ ⊆ A. By using (2), αcl(A) ∩ X] $_{gsc}$ ⊆]GSO-ker(A) ∩ X] $_{gsc}$ ⊆ A.

(ii) It is obtained by (2).

(3) ⇒ (2) By lemma 2.16 and (3),

αcl(A) = αcl(A) ∩ X = αcl(A) ∩ (X] $_{gsc}$ ∪ X$_e$ o) $_{gα}$
= (αcl(A) ∩ X] $_{gsc}$) ∪ (αcl(A) ∩ X$_e$)
= A ∪] GSO-ker(A) $_{gαo}$
=]GSO-ker(A) holds.

Theorem 2.18. Let (X, τ) be a space and A and B are subsets.
(i) If A is]gs-open and e gα -closed, then A is α-closed in (X, τ).
(ii) Suppose that (X, τ) is an α-space. A gα-closed set A is α-closed in (X, τ)^e if and only if αcl(A) − A is α-closed in (X, τ).
(iii) For each x ∈ X , {x} is]gs-closed or X − {x} is e gα -closed in (X, τ).
(iv) Every subset is gα-closed in (X, τ) if and only if]gs-open set is α-closed.^{e}

Proof. (ii) (Necessity) If A is α-closed, then αcl(A) − A = φ. αcl(A) − A is α-closed. It gα-closed and in follows from assumptions that τ = τ ^{α}. Then, αcl(A) − A is]gs-closed (X, τ) and by Theorem 2.14., αcl(A) − A = φ. Therefore A is α-closed in (X, τ).

(iii) If {x} is not is]gs-closed, then X − {x} is not]gs-open. Therefore X − {x} gα-closed in (X, τ).^{e}

(iv) (Necessity) Let U be a]gs-open set. Then we have that αcl(U) ⊆ U and hence U is α-closed. (Sufficiency) Let A be a subset and U is a]gs-open set such that A ⊆ U . Then αcl(A) αcl(U) = U and hence A is gα-closed. e

Remark 2.19. The following diagram shows the relationships established be- tween gα-closed sets and some other sets. A → B represents A implies B but note conversely.

α − closed gs − closed gα − closed
& ↑ %
closed −→ gα − closed −→ gsp − closed
. ↓ &
semi − preclosed sg − closed gpr − closed

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